Tutorial 8 2022.11.23

8.1 Shoelace formula

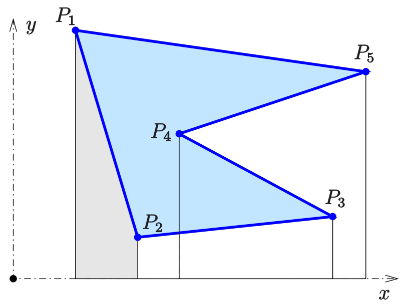

Figure 8.1: A 5-gon

Using Green's theorem, we have a formula to compute the area of a polygon in the plane $\mathbb{R}^2$ if we know the coordinates of the vertices.

A planar simple polygon P is represented by a positively oriented (counter clock wise) sequence of points $P_i = (x_i, y_i)\,, i = 1, \ldots, n$ in the Cartesian coordinate system. For the simplicity of the formulas below it is convenient to set $P_0 = P_n, P_{n+1} = P_1$. Figure 8.1 is an example of a polygon for $n = 5$. Suppose the boundary of P is denoted by ∂P which consists of straight line segments $\overline{P_i P_{i+1}}$.

Theorem 8.1 (Shoelace formula)

$$|P| = \frac{1}{2}\sum_{i=1}^{n}(x_i y_{i+1} - x_{i+1} y_i) = \frac{1}{2}\sum_{i=1}^{n}\begin{vmatrix} x_i & x_{i+1} \\ y_i & y_{i+1} \end{vmatrix}$$

♡

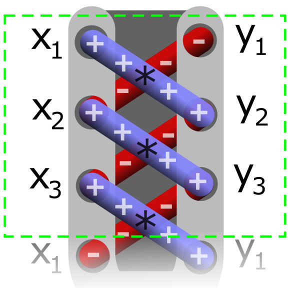

Figure 8.2: Shoelace scheme

Proof The area of P is $|P| = \iint_P 1 dxdy$. By Green's theorem,

$$|P| = \iint_P 1 dxdy = \frac{1}{2}\oint_{\partial P} xdy - ydx = \frac{1}{2}\sum_{i=1}^{n}\int_{\overline{P_i P_{i+1}}} xdy - ydx$$

Let $c : [0,1] \to \overline{P_i P_{i+1}}$ by $c(t) = (x_i + (x_{i+1} - x_i)t, y_i + (y_{i+1} - y_i)t)$ be a parametrization of the segment $\overline{P_iP_{i+1}}$. Then we have

$$\begin{aligned}
&\int_{\overline{P_iP_{i+1}}} xdy - ydx \\
=& \int_0^1 (x_i + (x_{i+1} - x_i)\, t)\,(y_{i+1} - y_i)\, dt - \int_0^1 (y_i + (y_{i+1} - y_i)\, t)\,(x_{i+1} - x_i)\, dt \\
=& \int_0^1 (x_i y_{i+1} - y_i x_{i+1})\, dt \\
=& \det \begin{pmatrix} x_i & x_{i+1} \\ y_i & y_{i+1} \end{pmatrix}
\end{aligned}$$

Therefore

$$|P| = \frac{1}{2} \sum_{i=1}^{n} (x_i y_{i+1} - x_{i+1} y_i) = \frac{1}{2} \sum_{i=1}^{n} \begin{vmatrix} x_i & x_{i+1} \\ y_i & y_{i+1} \end{vmatrix}$$

Remark It is hard to generalize the proof to high dimensions to compute volume of polytopes. Given a n-dimension polytope P in $\mathbb{R}^n$. Then Stokes's theorem implies

$$|P| = \int_P 1 dx_1 dx_2 \cdots dx_n = \int_{\partial P} x_1 dx_2 \cdots dx_n$$

Although the computation is reduced to dimension $n-1$, it is still complicated to compute.

On the other hand, there is a direct way via linear algebra. We may assume the facets of P is the union of simplexes by triangulation. Firstly we could compute the volume of a simplex Δ_n spanned by $n+1$ points $P_i = (x_i^1, x_i^2, \cdots, x_i^n), i = 1, \cdots, n+1$ using determinants.

For convenience, let's assume $n = 3$, then

$$\begin{aligned}
\mid \Delta_3 \mid =& \mid \det \begin{pmatrix} x_2^1 - x_1^1 & x_3^1 - x_1^1 & x_4^1 - x_1^1 \\ x_2^2 - x_1^2 & x_3^2 - x_1^2 & x_4^2 - x_1^2 \\ x_2^3 - x_1^3 & x_3^3 - x_1^3 & x_4^3 - x_1^3 \end{pmatrix} \mid \\
=& \mid \det (P_2P_3P_4) + \det (P_3P_1P_4) + \det (P_1P_2P_4) + \det (P_2P_1P_3) \mid .
\end{aligned}$$

Here $\det (P_iP_jP_k) = \det \begin{pmatrix} x_i^1 & x_j^1 & x_k^1 \\ x_i^2 & x_j^2 & x_k^2 \\ x_i^3 & x_j^3 & x_k^3 \end{pmatrix}$. The second equality is by multi-linearity of determinant.

By cancellation, the determinant about a common facet of two simplexes does not contribute. So if we assume P is a polytope whose facets are all simplexes, then $|P| = |\sum \det (P_{i_1} P_{i_2} \cdots P_{i_n})|$. The summation is taken over sequences of n vertices that form a facet of P in the order that has a compatible orientation.

8.2 Isoperimetric inequality

Theorem 8.2 (The Isoperimetric Inequality)

Let $c(t) = (x(t), y(t)), t \in [0,1]$ be a simple, closed, positively oriented and regular parameterised curve with $\mathrm{t} \in [a, b]$. *Denote the area enclosed in the above defined curve* $\mathrm{c}(\mathrm{t})$ *with A. Denote the length* *of* c(t) *by* $l := \int_0^1 \sqrt{x'(t)^2 + y'(t)^2} dt$*, we then have*

$$A \leq \frac{l^2}{4\pi}$$

with equality iff c(t) *is a circle.* ♡

The theorem comes from the question: Among all closed curves in the plane of fixed perimeter, which curve (if any) maximizes the area of its enclosed region?

Proof

The function $x(t)$ must be bounded. Say $m = \max_{t\in[0,1]} |x(t)|$. We may assume domain bounded by the curve c is convex, and by horizontal shifting we may assume $x'(t) > 0, 0 < t < p; x'(t) > 0, p < t < 1, x(0) = -m, x(p) = m$.

Define a circle by the parametrization $k(t) = (x(t), z(t),\ z(t) = -\sqrt{m^2 - x(t)^2}$ for $0 \leq t < p$ and $z(t) = \sqrt{m^2 - x(t)^2}$ for $p \leq t \leq 1$.

The area $A = \oint_c x dy = \int_0^1 x(t)y'(t)dt$. Let B be the area enclosed by $k(t)$. Then $B = \int_0^1 x(t)z'(t)dt = -\int_0^1 z(t)x'(t)dt = \pi m^2$. Add A to B,

$$\begin{aligned} A + B = A + \pi m^2 &= \int_0^1 \left(y'x - zx'\right) dt \\ &\leq \int_0^1 \sqrt{(y'x - zx')^2} dt \\ &\leq \int_0^1 \sqrt{(x^2 + z^2)\left((x')^2 + (y')^2\right)} dt \\ &= \int_0^1 m\sqrt{x'(t)^2 + y'(t)^2} dt = lm \end{aligned}$$

By mean inquality,

$$\begin{aligned} &\sqrt{A}\sqrt{\pi m^2} \leq \frac{A + \pi m^2}{2} \leq \frac{lm}{2} \\ &\Rightarrow A \leq \frac{l^2}{4\pi} \end{aligned}$$

To get equality, we have $A = \pi m^2 = \frac{1}{2} lm$ and $-xx' = zy'$ for all the inequality above. Squaring we get $x^2(x'^2 + y'^2) = m^2 y'^2$. We may assume $x'^2 + y'^2 = l^2$ by choosing a different parametrization. Thus $2\pi x = \pm y'$. Exchanging the role of x and y we got $2\pi y = \pm x'$. Finally

$$x^2 + y^2 = \frac{1}{4\pi^2}(x'^2 + y'^2) = \frac{l^2}{4\pi^2} = m^2$$

So $c(t)$ is a circle of radius m.

8.3 Area of surface of revolution

Problem 8.1 Let S be the surface of revolution obtained by rotating $\mathbf{r}(t) = (f(z), z), f(z) > 0, z \in [a, b]$ around the z-axis. Show that its surface area is given by

$$2\pi \int_a^b f(z)\sqrt{1 + f'^2(z)} dz.$$

Derive this formula using Riemann sum approach.